The Lee Fields Medal IV: SOLUTIONS

Time Allowed: Up to Two Hours and 15 Minutes

Tables and calculators may be used.

1. What is the last digit of 7^{2022}?

   Solution: There are various ways to determine the last digit is <u>nine</u>. If you simply play around with powers of seven you will see there is a periodicity to the last digit and that is how many people will get the final answer, and simply giving the final digit will suffice for full marks. Let us prove slightly more carefully that the last digit is indeed nine.

   Let us first examine some powers of 7:

   $$7^1 = 7,\ 7^2 = 49,\ 7^3 = 343,\ 7^4 = 2,401.$$

   So we have that $7^4 = 10m_1 + 1$ where $m_1 = 240$, but actually all that is important is that $m_1 \in \mathbb{N}$. Now what happens if we take powers of 7^4:

   $$(7^4)^2 = (10m_1 + 1)^2 = 100m_1^2 + 20m_1 + 1.$$

   Note that the first two terms are multiples of ten and so $(7^4)^2 = 10m_2 + 1$ for some $m_2 \in \mathbb{N}$. Now consider:

   $$(7^4)^3 = (7^4)^2 \times 7^4 = (10m_2 + 1)(10m_1 + 1) = 100m_1m_2 + 10m_1 + 10m_2 + 1.$$

   Again the first few terms are all multiples of ten and so $(7^4)^3 = 10m_3 + 1$ for some $m_3 \in \mathbb{N}$. Now suppose that $(7^4)^k = 10m_k + 1$ for some $m_k \in \mathbb{N}$. Then

   $$(7^4)^{k+1} = (7^4)^k(7^4) = (10m_k+1)(10m_1+1) = 100m_km_1+10m_k+10m_1+1 = 10m_{k+1}+1$$

   for some $m_{k+1} \in \mathbb{N}$. By induction, any power $(7^4)^n = 10m_n + 1$ for some $m_n \in \mathbb{N}$.

   Now look at

   $$7^{2022} = 7^{2020} \cdot 7^2 = (7^4)^{505}7^2 = (10m_{505} + 1)49 = 490m_{505} + 49 = 490m_{505} + 40 + 9,$$

   which has last digit nine, as promised.

2. Describe a way of generating better and better approximations to $2^{\sqrt{2}}$ using surds.

Solution: This question requires knowing three pieces of information:

(a) That real numbers can be approximated by terminating decimals,
(b) That terminating decimals are fractions,
(c) That a fractional power is a surd.

So we start with a calculator approximation to $\sqrt{2}$:

$$\sqrt{2} \approx 1.41421\ldots.$$

Now we can *truncate* the decimal approximation to generate a sequence of approximations to $\sqrt{2}$:

$$1.4,\ 1.41,\ 1.414,\ 1.4142,\ldots.$$

Each of these terminating decimals are fractions:

$$\frac{14}{10}, \frac{141}{100}, \frac{1414}{1000}, \frac{14142}{10000}, \ldots$$

If we want we can write these in reduced form (i.e. $14/10 = 7/5$, etc.). Now we note that:

$$a^{m/n} = \sqrt[n]{a^m},$$

and so we have surd approximations to $2^{\sqrt{2}}$:

$$\sqrt[10]{2^{14}},\ \sqrt[100]{2^{141}},\ \sqrt[1000]{2^{1414}},\ \sqrt[10000]{2^{14142}},\ldots.$$

This description suffices for full marks. What about a proof that these are actually better and better approximations to $2^{\sqrt{2}}$? Let $q_k = m_k/n_k$ be the k-th approximation to $\sqrt{2}$ as a fraction,

$$\sqrt{2} \approx \frac{m_k}{n_k},$$

e.g. $q_4 = \frac{14142}{10000}$. Now, what is the maximum difference between q_k and $\sqrt{2}$?

$$\begin{aligned}
\sqrt{2} &= \frac{14}{10} + 0.01421\ldots \implies |\sqrt{2} - q_1| \leq 0.1 = \frac{1}{10^1} \\
\sqrt{2} &= \frac{141}{100} + 0.00421\ldots \implies |\sqrt{2} - q_2| \leq 0.01 = \frac{1}{10^2} \\
\sqrt{2} &= \frac{1414}{1000} + 0.00021\ldots \implies |\sqrt{2} - q_3| \leq 0.001 = \frac{1}{10^3} \\
\sqrt{2} &= \frac{14142}{10000} + 0.00001\ldots \implies |\sqrt{2} - q_4| \leq 0.0001 = \frac{1}{10^4}
\end{aligned}$$

and similarly $|\sqrt{2} - q_k| \leq \frac{1}{10^k}$. In any case we can write:

$$q_k = \sqrt{2} + \varepsilon_k \qquad (|\varepsilon_k| \leq 1/10^k).$$

Therefore:

$$|2^{\sqrt{2}} - 2^{q_k}| = |2^{\sqrt{2}} - 2^{\sqrt{2}+\varepsilon_k}| = |2^{\sqrt{2}} - 2^{\sqrt{2}}2^{\varepsilon_k}| = |2^{\sqrt{2}}(1 - 2^{\varepsilon_k})| = |2^{\sqrt{2}}||1 - 2^{\varepsilon_k}|.$$

As $k \to \infty$, $\varepsilon_k \to 0$ and so $2^{\varepsilon_k} \to 1$ and so:

$$|2^{\sqrt{2}} - 2^{q_k}| \to 0 \implies 2^{q_k} \to 2^{\sqrt{2}}.$$

3. Take an odd number $n \geq 3$ and square it. Then divide it by two and look at the two integers closest to what you get, say b and c.

   Show that no matter what odd number n you choose, the numbers (n, b, c) are the side lengths of a right-angled-triangle.

   For example, let $n = 3$ so that $n^2 = 9$. This number divided by two is 4.5, with nearest integers $b = 4$ and $c = 5$, and there is a right-angled triangle with side-lengths 3, 4, 5.

   Solution: The trick here is to write $n = 2k + 1$ for some $k \in \mathbb{N}$. From here we square:

$$n^2 = (2k+1)^2 = 4k^2 + 4k + 1,$$

   halve:

$$\frac{n^2}{2} = 2k^2 + 2k + \frac{1}{2},$$

   and so we know the two whole numbers closest to this are $b = 2k^2 + 2k$ and $c = 2k^2 + 2k + 1$. So the proposal is that we have a triangle with side-lengths:

$$n = 2k + 1,\ b = 2k^2 + 2k,\ c = 2k^2 + 2k + 1.$$

   To show that in fact there is a right-angled triangle with these side-lengths we just have to use the converse of Pythagoras Theorem which says that if a triangle has side-lengths n, b, c such that $n^2 + b^2 = c^2$, then the triangle is right-angled.

   When $k \in \mathbb{N}$, $2k^2 > 1$ and so $2k^2 + 2k > 2k + 1$ so we have that:

$$n < b < c.$$

   Let us look at the square of the longest side:

$$\begin{aligned} c^2 = (2k^2 + 2k + 1)^2 &= ((2k^2 + 2k) + 1)^2 \\ &= (2k^2 + 2k)^2 + 2(2k^2 + 2k) + 1 \\ &= (2k^2 + 2k)^2 + 4k^2 + 4k + 1 \\ &= (2k^2 + 2k)^2 + (2k + 1)^2 = n^2 + b^2, \end{aligned}$$

   that is the sum of the squares of the other two sides. We have a right-angles triangle with side-lengths n, b, c as required.

4. Find the equation of the parabola(s) $y = ax^2 + bx + c$ which contains the points $P(1, 0)$, $Q(2, 3)$, $R(3, 10)$.

   Solution: A point (x_0, y_0) is on a curve if and only if x_0, y_0 satisfies the equation of the curve. So we have[1]:

$$\begin{aligned} a(1)^2 + b(1) + c = 0 &\implies a + b + c = 0 \\ a(2)^2 + b(2) + c = 3 &\implies 4a + 2b + c = 3 \\ a(3)^2 + b(3) + c = 10 &\implies 9a + 3b + c = 10. \end{aligned}$$

[1] simultaneous equations in three unknowns are *not* on the ordinary leaving cert syllabus and we apologise for this oversight.

We can find in each case c in terms of a and b:

$$c = -a - b$$
$$c = 3 - 4a - 2b$$
$$c = 10 - 9a - 3b,$$

and setting these equal to each other we generate equations in a and b:

$$-a - b = 3 - 4a - 2b \implies 3a + b = 3$$
$$3 - 4a - 2b = 10 - 9a - 3b \implies 5a + b = 7$$

We can find $b = 3 - 3a$ and $b = 7 - 5a$:

$$\implies 3 - 3a = 7 - 5a \implies 2a = 4 \implies a = 2,$$

and from here $b = 3 - 3(2) = -3$ and $c = -2 - (-3) = 1$ and so:

$$y = 2x^2 - 3x + 1:$$

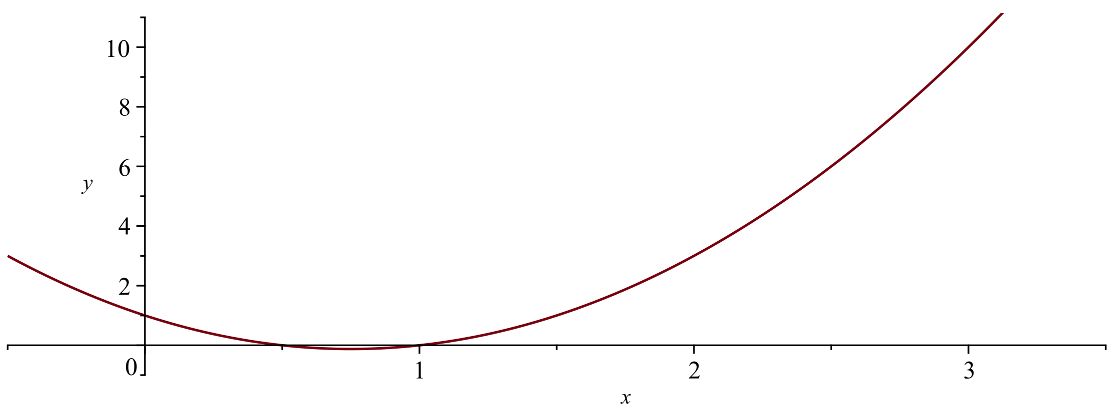
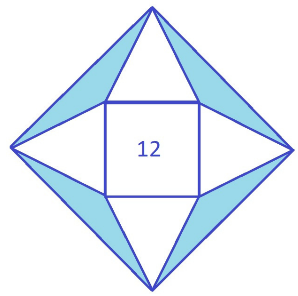
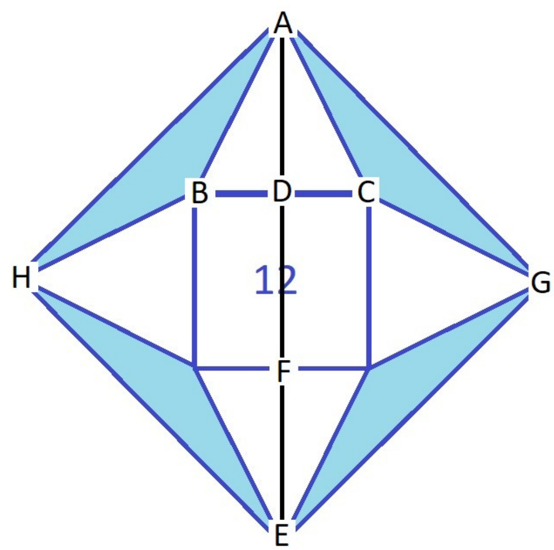


Figure 1: Here we can see the parabola passing through the three points, $(0, 1)$, $(2, 3)$, $(3, 10)$.

5. Four equilateral triangles are arranged around a square which has area 12. What's the total shaded area?



Solution: Label as follows:



Let the square of area 12 have side-length s so that:

$$s^2 = 12 \implies s = \pm\sqrt{12} \underset{s>0}{=} 2\sqrt{3} \implies |BD| = \frac{1}{2}2\sqrt{3} = \sqrt{3}.$$

As an equilateral triangle, $\triangle ABC$ has all angles $60°$, in particular $\angle ABC$. Note

$$\tan(\angle ABC) = \frac{|AD|}{|BD|} \implies |AD| = |BD|\tan(60°) = \sqrt{3}\sqrt{3} = 3.$$

By symmetry $|FE| = |AD| = 3$. Therefore

$$|AE| = |AD| + |DF| + |FE| = 3 + 2\sqrt{3} + 3 = 6 + 2\sqrt{3}.$$

Let $|AG| = |GE| = a$ be the side-length of $\square AGEH$. As a right-angled triangle ΔAGE, Pythagoras speaks:

$$|AE|^2 = |AG|^2 + |GE|^2 \implies (6 + 2\sqrt{3})^2 = 2a^2 \implies a = \frac{6 + 2\sqrt{3}}{\sqrt{2}}.$$

So now we know that $A(\square AGEH) = a^2 = (6 + 2\sqrt{3})^2/2 = 24 + 12\sqrt{3}$.

The area of each of the triangles is:

$$A(\Delta) = \frac{1}{2}bh = \frac{1}{2}2\sqrt{3} \times 3 = 3\sqrt{3}$$

and so the area of the shaded region is:

$$A(\square AGEH) - 4 \cdot A(\Delta) - A(\square \text{ of area } 12) = 24 + 12\sqrt{3} - 4 \cdot 3\sqrt{3} - 12 = 12.$$

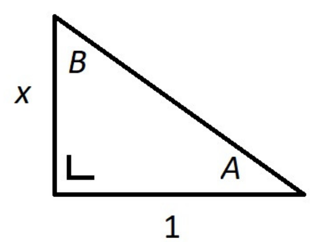
This is a somewhat *Pavlovian* solution, reacting without great thought to the problem. A number of students had the following, far more elegant solution. Look again at ΔACG. Note it is isosceles triangle ($|AC| = |CG|$), and also this length $|AC| = |BC| = \sqrt{12}$. Note also that the known angles at C are $60°$, $60°$, and $90°$, which sum to $210°$, and so $\angle ACG = 360° - 210° = 150°$. From here

$$A(\Delta ACG) = \frac{1}{2}ab\sin C = \frac{1}{2}\sqrt{12}\sqrt{12}\sin(150°) = 3 \implies A(\text{shaded}) = 4 \times 3 = 12.$$

6. With the aid of a diagram, explain why for $x > 0$:

$$\tan^{-1}(x) + \tan^{-1}\left(\frac{1}{x}\right) = 90°.$$

Solution: Draw a right-angled triangle with side-lengths one and x:



We have $\tan A = x/1 = x \implies A = \tan^{-1}(x)$ and similarly:

$$\tan B = \frac{1}{x} \implies B = \tan^{-1}\left(\frac{1}{x}\right),$$

and so

$$\tan^{-1}(x) + \tan^{-1}\left(\frac{1}{x}\right) = A + B = 90°,$$

as the angle measures must sum to $180°$.

7. Six cups and saucers come in pairs: there are two cups and saucers which are red, two white, and two black. If the cups are randomly placed onto the saucers (one each), find the probability that no cup is on a saucer of the same colour.

   Solution: How many ways can the cups be placed on the saucers? Well, for the two red cups, we must from six saucers choose two, this can be done:

   $$\binom{6}{2} = 15 \text{ ways.}$$

   Then for the white cups, we must from four remaining saucers choose two, this can be done:

   $$\binom{4}{2} = 6 \text{ ways.}$$

   Then there are no remaining choices for the black cups, so in total there are:

   $$\binom{6}{2}\binom{4}{2} = 90 \text{ ways}$$

   of placing the cups on the saucers.

   Now, how can we place no cup on a saucer of the same colour. We can place both the red on the white, both the white on the black, and both the black on the red. This is one way. Alternatively we can place both the red on the black, both the white on the red, and both the black on the white. This is a second possibility.

   Alternatively, we can have one red on each of white and black, one white on each of red and black, and one black on each of white and red. The white saucers can be (R, B) or (B, R), the black saucers can be (W, R) or (R, W), and the red saucers can be (W, B) or (B, W), that is there are:

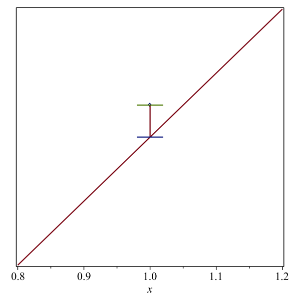
   $$2 \times 2 \times 2 = 8 \text{ ways}$$

   where we don't have e.g. both red on white.

   Therefore,

   $$\mathbb{P}[\text{no cup on same colour}] = \frac{1+1+8}{90} = \frac{1}{9}.$$

8. Consider three points $(1, 2.1), (2, 3.8), (3, 6.1)$ on the plane. Find the line $y = mx$ that minimises the sum of the *squared deviations* of the points from the line.



Zooming in near a point (x_i, y_i), we see its deviation from the line $y = mx$. This deviation is the vertical distance from the point to the line, and the *squared deviation* of (x_i, y_i) from $y = mx$ is equal to $(mx_i - y_i)^2$. In the case of the point $(1, 2.1)$, this squared deviation is $(m(1) - 2.1)^2 = (m - 2.1)^2$.

Solution: We have that the sum of the squared deviations is:

$$s(m) = (m - 2.1)^2 + (2m - 3.8)^2 + (3m - 6.1)^2.$$

This can be multiplied out to get $s(m) = 14m^2 - 56m + 56.06$. Now we want to find the minimum and this occurs where $s'(m) = 0$, so we want to find the m such that:

$$\begin{aligned} s'(m) &= 0 \\ \implies 14 \cdot 2m - 56(1) &= 0 \\ \implies 28m &= 56 \\ \implies m &= 2. \end{aligned}$$

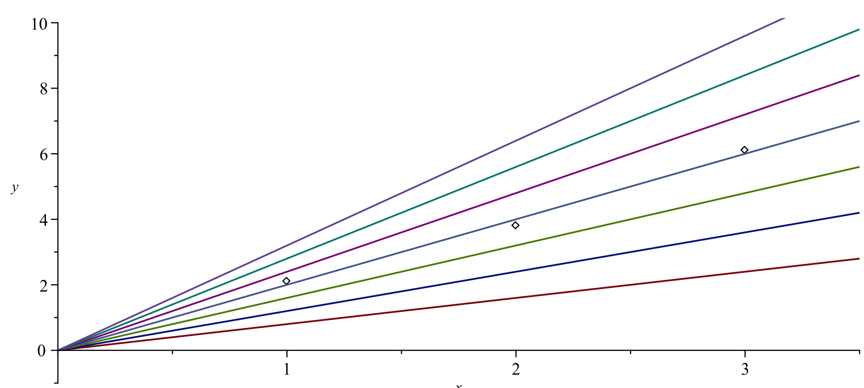
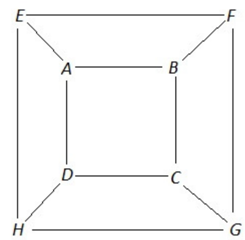
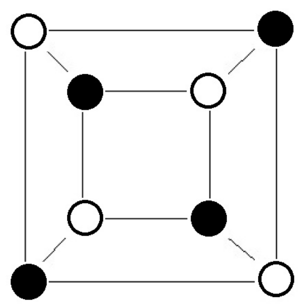


Figure 2: Here we see the lines $y = 0.8x$, $y = 1.2x$, $y = 1.6x$, $y = 2x$, $y = 2.4x$, $y = 2.8x$ and $y = 3.2x$. Not shown: infinitely many more lines, one for each $m \in \mathbb{R}$. Out of all these infinitely many lines, the line $y = 2x$ is the best fit to the points in this, *least squares*, sense.

9. Johnny & Mary live in a town with only eight pubs but they still want to do the 12 Pubs of Christmas. They decide to start in Pub A and after each drink move to one of the three adjacent pubs whether they have been there before or not. If they do *not* have to visit every pub, can Johnny & Mary end up back in Pub A for their 12th drink? Justify your answer.



Solution: This is an example of a *bipartite graph*. That is the nodes $A, B, \ldots, H$ can be coloured black and white, such that no black node is adjacent to any other black node, and similarly for the white nodes:



Now, pub A is a black node, and the point is that whenever John and Mary are in a black node, the next node is white. And whenever they are in a white node, the next node is black. So starting at pub A, black, the next 11 pubs will be:

$$W \to B \to W \to B \to W \to B \to W \to B \to W \to B \to W.$$

In other words, the even pubs are white and the odds pubs are black. So on the 12th drink they are in a white pub, and A is not a white pub, so they must cannot end up there on their final drink.

10. Two cubes are used in a calendar to display the day for the current month as shown below. List what should be written on the sides of each cube so that all days 1-31 can be displayed by the calendar:

Solution: The greedy approach is to put $0, \ldots, 5$ on the first cube. Then you need at least $6, 7, 8, 9$ on the second cube, together with $1, 2$ for the 11th and 22nd days... but this misses out on 03, 04, 05. The trick is to realise that 6 can do 6 and 9. So if you have $0, \ldots, 5$ on the first cube, and $1, 2, 6, 7, 8$ on the second, you no longer need to miss out on 03, 04, 05, because you can put a second zero on the second cube. Now you have everything, by inspection, with:

$$\{0, 1, 2, 3, 4, 5\} \text{ and } \{0, 1, 2, 6, 7, 8\}$$